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COMMENT

Comment on Parisi's equation for the SK model for spin glasses

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Abstract. A simple algebraic derivation of Parisi's equations for spin glasses is given.

Parisi (1980) has recently proposed a solution for the Sherrington–Kirkpatrick model of spin glasses, which involves a function $q(x)$, $x \in [0, 1]$, as a local order parameter. The free-energy density is then given by the nonlinear differential equation

$$\frac{\partial f}{\partial x} = -\frac{1}{2}\frac{\mathrm{d}q}{\mathrm{d}x}\left[\frac{\partial^2 f}{\partial h^2} + x\left(\frac{\partial f}{\partial h}\right)^2\right] \tag{1}$$

where h is the external magnetic field. $f(x, h)$ is a generalised free energy such that $f(1, h) = \ln(2\cosh h)$. $f(0, h)$ gives the actual free energy.

Starting from Parisi's form for the continuous breaking of the replica symmetry, one can give by a source field method a simple algebraic derivation of (1).

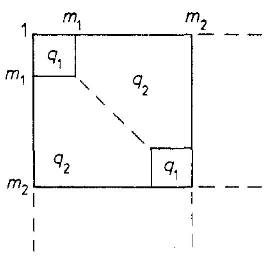
The quantity of interest is

$$F = -\lim_{n\to 0}\frac{1}{n}\ln G, \tag{2}$$

$$G = \sum_{S_a=\pm 1}\exp\left(\frac{1}{2}\sum_{a=1}^{n}\sum_{b=1}^{n} Q_{ab}S_aS_b + h\sum_{a=1}^{n} S_a\right). \tag{3}$$

Q_{ab} is a $n \times n$ matrix. Then Parisi considers the following parametrisation for Q:

$Q =$



(4)

where the $m_i (i = 1, K)$ are the successive sizes of the diagonal blocks, with $1 \leqslant m_1 \leqslant \ldots \leqslant m_K \leqslant m_{K+1} = n$. $q(x)$ is defined by $q(m_i) = q_i$.

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Introducing a source h_a, we write G as

$$G = \sum_{S_a=\pm 1} \left(\exp \frac{1}{2} \sum_{a,b} Q_{ab} \frac{\partial}{\partial h_a} \frac{\partial}{\partial h_b}\right) \exp \sum_a h_a S_a \bigg|_{h_a=h}$$
$$= \left(\exp \frac{1}{2} \sum_{a,b} Q_{ab} \frac{\partial}{\partial h_a} \frac{\partial}{\partial h_b}\right) \left(\prod_a 2 \cosh h_a\right)\bigg|_{h_a=h}. \tag{5}$$

For the trivial case $Q_{ab} = q$,

$$G = \left(\exp \frac{1}{2} q \sum_{a,b} \frac{\partial}{\partial h_a} \frac{\partial}{\partial h_b}\right) \left(\prod_a 2 \cosh h_a\right)\bigg|_{h_a=h}$$
$$= \left(\exp \frac{1}{2} q \frac{\partial^2}{\partial h^2}\right) (2 \cosh h)^n \tag{6}$$

where we have used repeatedly the trivial identity $\Sigma_a \, \partial f(h_1, \ldots, h_n)/\partial h_a|_{h_a=h} = \partial f(h, \ldots, h)/\partial h$. Equation (6) is the key to the calculation of G. The generic matrix Q can indeed be considered as the limit of the series

$$Q = \lim_{mi \to m_K} \left\{\tilde{q}_1 \boxed{\;1\;}^{\,m_1}, \tilde{q}_1 \begin{pmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \end{pmatrix}^{m_1 \; m_2} + \tilde{q}_2 \boxed{\;1\;}^{\,m_2}, \ldots\right\} + q(n) \boxed{\;1\;}^{\,n},$$

where the $\tilde{q}_i$ are defined by

$$\tilde{q}_i = q_i - q_{i+1} = q(m_i) - q(m_{i+1}) \tag{8}$$

and characterise the superposed sheets of the diagonal blocks. $\boxed{1}^m \equiv I_m$ is the constant $m \times m$ matrix equal to 1 everywhere. Then (7) reads symbolically

$$Q = \lim_{m_i \to m_K} \{\tilde{q}_1 I_{m_1}, \tilde{q}_1 (I_{m_1})^{[m_2/m_1]} + \tilde{q}_2 I_{m_2}, \ldots\} + q(n) I_n. \tag{9}$$

Now $g(m_i, h)$ is defined as the restricted $G(5)$ calculated for the i-th term of the series (7). (9) (6) give immediately the recursion:

$$g(m_1, h) = (\exp \tfrac{1}{2}\tilde{q}_1 (\partial^2/\partial h^2))(2 \cosh h)^{m_1},$$
$$g(m_2, h) = (\exp \tfrac{1}{2}\tilde{q}_2 (\partial^2/\partial h^2))[g(m_1, h)]^{m_2/m_1} \ldots,$$
$$G = [\exp \tfrac{1}{2} q(n)(\partial^2/\partial h^2)][g(m_K, h)]^{n/m_K}.$$

In the continuous limit $n \to 0$, $m_i = x \in [0, 1]$, $m_{i+1}/m_i = (x + \mathrm{d}x)/x$, and the recursion relation becomes

$$g(x + \mathrm{d}x, h) = [\exp(-\tfrac{1}{2} \, \mathrm{d}q(x)(\partial^2/\partial h^2))]([g(x, h)]^{1 + \mathrm{d} \ln x}) \tag{10}$$

with $g(1, h) = 2 \cosh h$. Equation (10) is equivalent to

$$\frac{\partial g}{\partial x} = -\frac{1}{2} \frac{\mathrm{d}q}{\mathrm{d}x} \frac{\partial^2 g}{\partial h^2} + \frac{1}{x} g \ln g. \tag{11}$$

Using (2) and (9), we find for $n \to 0$

$$F = -\left[\exp \frac{1}{2} q(0) \frac{\partial^2}{\partial h^2}\right] \frac{1}{x} \ln g(x, h)\Big|_{x=m_K \to 0}.$$

Then, because of (10), the function $f(x, h) = (1/x) \ln g(x, h)$ verifies equation (1), QED.

Reference

Parisi G 1980 *J. Phys. A: Math. Gen.* **13** L115–21